OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) **4758/01**

Differential Equations

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Insert for Question 2 (inserted)
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Wednesday 27 January 2010
Afternoon

Duration: 1 hour 30 minutes

* 4 7 5 8 0 1 *

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- There is an **insert** for use in Question **2**.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 A particle is attached to a spring and suspended vertically from an oscillating platform. The vertical displacement, y, of the particle from a fixed point at time t is modelled by the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 6\frac{\mathrm{d}y}{\mathrm{d}t} + 9y = 0.5\sin t.$$

(i) Find the general solution. **[9]**

Initially the displacement and velocity are both zero.

(ii) Find the solution. **[5]**

(iii) Describe the motion of the particle for large values of t. **[2]**

(iv) Find approximate values of the velocity and displacement at $t = 20\pi$. **[3]**

The motion of the platform is stopped at $t = 20\pi$ and the differential equation modelling the subsequent motion of the particle is

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 6\frac{\mathrm{d}y}{\mathrm{d}t} + 9y = 0.$$

(v) Write down the general solution. Sketch the solution curve for $t > 20\pi$. **[5]**

2 **There is an insert for use with part (b)(i) of this question.**

(a) The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = \tan x$$

is to be solved for $|x| < \frac{1}{2}\pi$.

(i) Find the general solution. **[8]**

(ii) Find the equation of the solution curve that passes through the origin and sketch the curve. **[4]**

(b) The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y^2\tan x = \tan x$$

is to be solved approximately, first by using a tangent field and then by Euler's method.

(i) On the insert is a tangent field for the differential equation. Sketch the solution curves through the origin and through $(0, 1)$. **[4]**

Euler's method is now used, starting at $x = 0$, $y = 1$. The algorithm is given by $x_{r+1} = x_r + h$, $y_{r+1} = y_r + hy'_r$.

(ii) Carry out two steps with a step length of 0.1 to verify that the algorithm gives $x = 0.2$, $y \approx 1.0201$. **[5]**

(iii) Explain why it would be inappropriate to extend this numerical solution as far as $x = 1.6$. **[2]**

(iv) How could the accuracy of the estimate found in part **(b)(ii)** be improved? **[1]**

3 Fig. 3 shows a small ball projected from a point O over horizontal ground. The forces acting on the ball are its weight and air resistance. Its initial horizontal component of velocity is v_1 and its subsequent horizontal velocity $\dot{x}$ is modelled by the differential equation

$$\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = -k\dot{x},$$

where k is a positive constant.

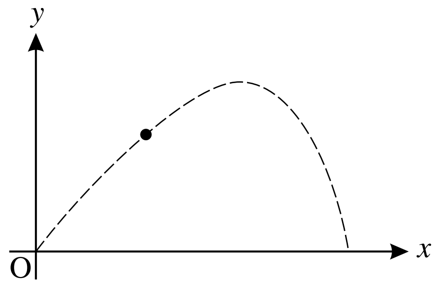

Fig. 3

The units of displacement are metres and the units of time are seconds.

(i) Solve this differential equation to find $\dot{x}$ in terms of t and hence show that the horizontal displacement from O is given by $x = \frac{v_1}{k}\left(1 - \mathrm{e}^{-kt}\right)$. **[8]**

The ball's initial vertical component of velocity is v_2 and its subsequent vertical velocity $\dot{y}$ is modelled by the differential equation

$$\frac{\mathrm{d}\dot{y}}{\mathrm{d}t} = -k\dot{y} - g.$$

(ii) Solve this differential equation to find $\dot{y}$ in terms of t and hence show that the vertical displacement from O is given by $y = \frac{kv_2 + g}{k^2}\left(1 - \mathrm{e}^{-kt}\right) - \frac{g}{k}t$. **[10]**

(iii) Eliminate t between the expressions for x and y to show that $y = \left(\frac{kv_2 + g}{kv_1}\right)x + \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_1}\right)$. **[4]**

(iv) In the case $v_1 = v_2 = 10$, $k = 0.1$, determine whether the ball will pass over a 5 m high wall at a horizontal distance 8 m from O. **[2]**

4 The simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x - 4y + 23,$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x + y - 7$$

are to be solved.

(i) Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 5$. **[5]**

(ii) Find the general solution for x. **[7]**

(iii) Find the corresponding general solution for y. **[4]**

When $t = 0$, $x = 8$ and $y = 0$.

(iv) Find the particular solutions for x and y. **[4]**

(v) Show that for sufficiently large t, y is always greater than x. **[4]**

THERE ARE NO QUESTIONS PRINTED ON THIS PAGE.

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MATHEMATICS (MEI) **4758/01**

Differential Equations

INSERT for Question 2

Wednesday 27 January 2010

Afternoon

Duration: 1 hour 30 minutes

Candidate Forename						Candidate Surname			
Centre Number						Candidate Number			

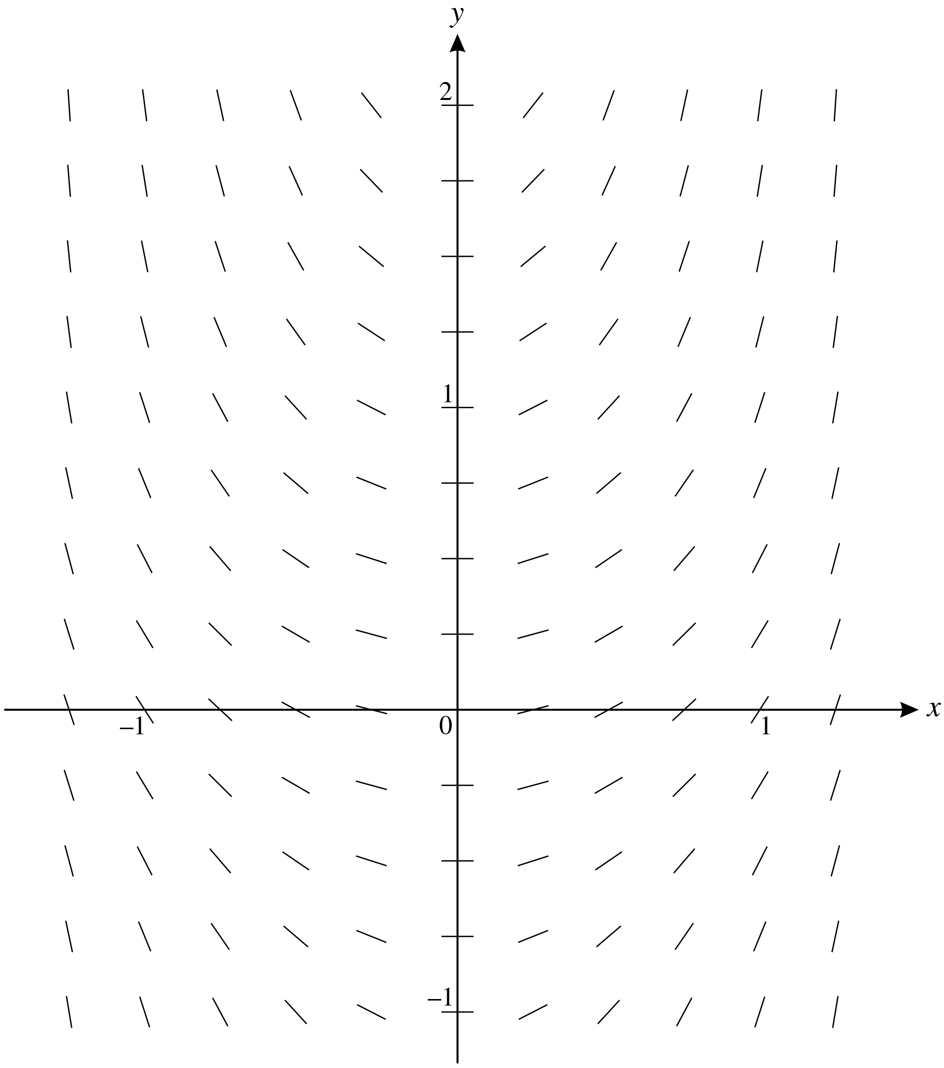
INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the boxes above.
- Use black ink. Pencil may be used for graphs and diagrams only.
- This insert should be used to answer Question **2** part **(b)(i)**.
- Write your answers to Question **2** part **(b)(i)** in the spaces provided in this insert, and **attach it to your Answer Booklet**.

INFORMATION FOR CANDIDATES

- This document consists of **2** pages. Any blank pages are indicated.

2 (b) (i)



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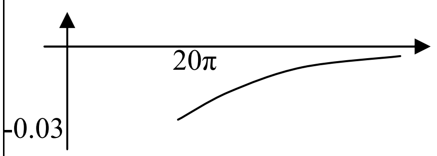
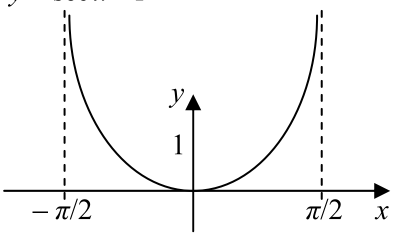
4758 Differential Equations

1(i)	$\alpha^2+6\alpha+9=0$ $\alpha=-3$ (repeated) $y=e^{-3t}(A+Bt)$ PI $y=a\sin t+b\cos t$ $\dot{y}=a\cos t-b\sin t$ $\ddot{y}=-a\sin t-b\cos t$ $-a\sin t-b\cos t+6(a\cos t-b\sin t)$ $+9(a\sin t+b\cos t)=0.5\sin t$ $8a-6b=0.5$ $8b+6a=0$ Solving gives $a=0.04, b=-0.03$ GS $y=e^{-3t}(A+Bt)+0.04\sin t-0.03\cos t$	**M1** **A1** **F1** **B1** **M1** **M1** **M1** **A1** **F1**	Auxiliary equation CF for their roots Differentiate twice and substitute Compare coefficients Solve PI + CF with two arbitrary constants	**9**
(ii)	$t=0, y=0 \Rightarrow A=0.03$ $\dot{y}=e^{-3t}(B-3A-3Bt)+0.04\cos t+0.03\sin t$ $t=0, \dot{y}=0 \Rightarrow 0=B-3A+0.04$ $y=0.01\left(e^{-3t}(3+5t)+4\sin t-3\cos t\right)$	**M1** **M1** **F1** **M1** **A1**	Use condition Differentiate Follows their GS Use condition Cao	**5**
(iii)	For large t, the particle oscillates With amplitude constant (≈ 0.05)	**B1** **B1**	Oscillates Amplitude approximately constant	**2**
(iv)	$t=20\pi \Rightarrow e^{-60\pi}$ very small $y\approx -0.03$ $\dot{y}\approx 0.04$	**M1** **A1** **A1**		**3**
(v)	$y=e^{-3t}(C+Dt)$ 	**M1** **A1** **B1√** **B1** **B1**	CF of correct type or same type as in (i) Must use new arbitrary constants $y\approx -0.03$ at $t=20\pi$ Gradient at 20π consistent with (iv) Shape consistent	**5**

2(a)(i)	$I = \exp\int -\tan x\,\mathrm{d}x$	**M1**	Attempt IF	
	$= \exp(-\ln\sec x)$	**A1**	Correct IF	
	$= (\sec x)^{-1} = \cos x$	**A1**	Simplified	
	$\cos x\frac{\mathrm{d}y}{\mathrm{d}x} - y\sin x = \sin x$	**M1**	Multiply by IF	
	$\frac{\mathrm{d}}{\mathrm{d}x}(y\cos x) = \sin x$	**M1**	Recognise derivative	
		M1	Integrate	
	$y\cos x = -\cos x + A$	**A1**	RHS (including constant)	
	($y = A\sec x - 1$)	**A1**	LHS	**8**
		M1	Rearrange equation	
		A1		
	$\frac{\mathrm{d}y}{\mathrm{d}x} = (1+y)\tan x$	**M1**	Separate variables	
		A1		
	$\ln(1+y) = \ln\sec x + A$	**M1**	RHS	
		A1		
		M1	LHS	
		A1		**8**
(ii)	$x = 0, y = 0 \Rightarrow 0 = A - 1$	**M1**	Use condition	
	$y = \sec x - 1$	**A1**		
		**B1** **B1**	Shape and through origin Behaviour at $\pm\frac{1}{2}\pi$	**4**
(b)(i)		**M1**	Attempt one curve	
		A1	Reasonable attempt at one curve	
		M1	Attempt second curve	
		A1	Reasonable attempt at both curves	**4**
(ii)	$y' = (1+y^2)\tan x$	**M1**	Rearrange	
	$x = 0, y = 1 \Rightarrow y' = 0$			
	$y(0.1) = 1 + 0.1\times 0 = 1$	**M1**	Use of algorithm	
	$x = 0.1, y = 1 \Rightarrow y' = 0.201$	**A1**		
	$y(0.2) = 1 + 0.1\times 0.201$	**M1**	Use of algorithm for second step	
	$= 1.0201$	**E1**		**5**
(iii)	$\tan\frac{\pi}{2}$ undefined so cannot go past $\frac{\pi}{2}$	**M1**		
	So approximation cannot continue to $1.6 > \frac{\pi}{2}$	**A1**		**2**
(iv)	Reduce step length	**B1**		**1**

3(i)	$\dot{x} = Ae^{-kt}$	**M1** **A1**	Any valid method (or no method shown)	
	$t = 0, \dot{x} = v_1 \Rightarrow A = v_1$	**M1**	Use condition	
	$\dot{x} = v_1 e^{-kt}$	**A1**		
	$x = \int v_1 e^{-kt} \mathrm{d}t$	**M1**	Integrate	
	$= -\frac{v_1}{k} e^{-kt} + B$	**A1**		
	$t = 0, x = 0 \Rightarrow B = \frac{v_1}{k}$	**M1**	Use condition	
	$x = \frac{v_1}{k}\left(1 - e^{-kt}\right)$	**E1**		
				8
(ii)	$\int \frac{\mathrm{d}\dot{y}}{\dot{y} + g/k} = \int -k\mathrm{d}t$	**M1**	Separate and integrate	
	$\mathrm{In}\left(\dot{y} + \frac{g}{k}\right) = -kt + C$	**A1**	LHS	
		A1	RHS	
	$\dot{y} + \frac{g}{k} = De^{-kt}$	**M1**	Rearrange, dealing properly with constant	
	$t = 0, \dot{y} = v_2 \Rightarrow D = v_2 + \frac{g}{k}$	**M1**	Use condition	
	$\dot{y} = \left(v_2 + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}$	**A1**		
	$y = \int\left(\left(v_2 + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}\right)\mathrm{d}t$	**M1**	Integrate	
	$= -\frac{1}{k}\left(v_2 + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}t + E$	**A1**		
	$t = 0, y = 0 \Rightarrow 0 = -\frac{1}{k}\left(v_2 + \frac{g}{k}\right) + E$	**M1**	Use condition	
	$y = \frac{1}{k^2}\left(kv_2 + g\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t$	**E1**		
				1 **0**
(iii)	$1 - e^{-kt} = \frac{kx}{v_1}$	**M1**		
	$t = -\frac{1}{k}\ln\left(1 - \frac{kx}{v_1}\right)$	**A1**		
	$y = \left(\frac{kv_2 + g}{kv_1}\right)x + \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_1}\right)$	**M1** **E1**	Substitute Convincingly shown	
				4
(iv)	$x = 8 \Rightarrow y = 4.686$ Hence will not clear wall	**M1** **A1**		
				2

4(i)	$4y = -3x + 23 - \dot{x}$	**M1**	y or $4y$ in terms of $x, \dot{x}$	
	$4\dot{y} = -3\dot{x} - \ddot{x}$	**M1**	Differentiate	
	$\frac{1}{4}(-3\dot{x} - \ddot{x}) = 2x + \frac{1}{4}(-3x + 23 - \dot{x}) - 7$	**M1**	Substitute for y	
	$-3\dot{x} - \ddot{x} = 8x - 3x + 23 - \dot{x} - 28$	**M1**	Substitute for $\dot{y}$	
	$\Rightarrow \ddot{x} + 2\dot{x} + 5x = 5$	**E1**		
				5
(ii)	$\alpha^2 + 2\alpha + 5 = 0$	**M1**	Auxiliary equation	
	$\Rightarrow \alpha = -1 \pm 2i$	**A1**		
		M1	CF for complex roots	
	CF $\mathrm{e}^{-t}(A\cos 2t + B\sin 2t)$	**F1**	CF for their roots	
	PI $x = \frac{5}{5} = 1$	**B1**	Constant PI	
		B1	Correct PI	
	GS $x = 1 + \mathrm{e}^{-t}(A\cos 2t + B\sin 2t)$	**F1**	PI + CF with two arbitrary constants	
				7
(iii)	$y = \frac{1}{4}(-3x + 23 - \dot{x})$	**M1**		
	$= \frac{1}{4}\Big[-3 - 3\mathrm{e}^{-t}(A\cos 2t + B\sin 2t) + 23$ $+\mathrm{e}^{-t}(A\cos 2t + B\sin 2t)$	**M1**	Differentiate and substitute	
	$-\mathrm{e}^{-t}(-2A\sin 2t + 2B\cos 2t)\Big]$	**F1**	Expression for $\dot{x}$ follows their GS	
	$y = 5 - \frac{1}{2}\mathrm{e}^{-t}\left((A+B)\cos 2t + (B-A)\sin 2t\right)$	**A1**		
				4
(iv)	$t = 0, x = 8 \Rightarrow 1 + A = 8 \Rightarrow A = 7$	**M1**	Use condition	
	$t = 0, y = 0 \Rightarrow 5 - \frac{1}{2}(A + B) = 0 \Rightarrow B = 3$	**M1**	Use condition	
	$x = 1 + \mathrm{e}^{-t}(7\cos 2t + 3\sin 2t)$	**A1**		
	$y = 5 - \mathrm{e}^{-t}(5\cos 2t - 2\sin 2t)$	**A1**		
				4
(v)	For large t, e^{-t} tends to 0	**M1**		
	$y \to 5$	**B1**		
	$x \to 1$	**B1**		
	$\Rightarrow y > x$	**E1**	Complete argument	
				4

4758 Differential Equations (Written paper)

General Comments

The standard of work was generally good, with the majority of candidates demonstrating a clear understanding of the techniques required. There were, however, a significant number of arithmetic and algebraic slips which led to loss of marks. Almost all candidates opted for Questions 1 and 4, with Question 3 being the least popular choice. Those who did attempt Question 3 usually gained very high marks for their solutions.

There was a slight improvement in the standard of the graph sketching, compared with previous series, but candidates need to realise that in this paper any known or previously calculated information should be indicated on the sketch. Any particular features, such as the behaviour in approaching an asymptote, should be clearly and carefully shown. No further calculations are required.

Comments on Individual Questions

1 Second order differential equation

(i) The method required here was well known, but errors in solving the linear simultaneous equations obtained in finding the particular integral were seen from many candidates.

(ii) Most candidates earned the method marks, but the accuracy of their differentiation and arithmetic was not of a good standard.

(iii) A description involving oscillatory motion with constant amplitude was expected here. It was not sufficient to find an expression for y for large t, with no further comment.

(iv) Most candidates substituted $t = 20\pi$ in their expressions for y and $\mathrm{d}y/\mathrm{d}x$ and gained the method mark.

(v) The majority of candidates gave their complementary function from part **(i)** as the general solution to this differential equation, but failed to realise that different arbitrary constants were required. The graphs were very variable in quality. Most candidates recognised that the solution curve approached zero for large t. It was rare, however, to see correct use of the values found in part **(iii)** as the initial values of the function and its gradient.

2 First order differential equation

(a) (i) There were two methods available here. The minority of candidates who opted for separation of variables almost always gained full marks for their solutions. Those who opted for use of the integrating factor fared less well. Many omitted the minus sign in front of $\tan x$ and could make little progress with the integral subsequently obtained on the right hand side of the differential equation. Others failed to recognise $\cos x \tan x$ as being equivalent to $\sin x$ and so were unable to integrate successfully.

(ii) There were very few correct graphs, often because the general solution obtained in part **(i)** was incorrect.

(b) **(i)** The general idea of the solution curves was universally known, but the drawing of them was often carelessly executed. Sketches should not cross the gradient lines in the given tangent field and need to be drawn with care to obtain full marks.

(ii) Euler's method was well-known and the calculations usually correct, but candidates need to ensure that they show sufficient working to justify an answer that is given in the question.

(iii) The limits for x given in part **(a)** of this question were often assumed to apply in part **(ii)**. This led to answers that noted that 1.6 is greater than $\frac{1}{2}\pi$, rendering the method invalid. Marks were only awarded when candidates realised the existence of the asymptote at $\frac{1}{2}\pi$.

3 First order differential equations

This was the least popular question, with candidates perhaps deterred by its algebraic nature. The question was, however, structured to lead candidates through the process and those who attempted it usually scored very high marks. The given answers enabled them to work more accurately, as they very sensibly checked back through their work when they did not immediately attain the given answer.

4 Simultaneous differential equations

The vast majority of candidates knew the methods involved and if they had been able to execute them accurately, would have obtained at least 23 of the 24 available marks.

(i) Solutions were almost always fully correct, exhibiting an accuracy of work that seemed to evaporate for the remaining parts of the question.

(ii) The complementary function was found successfully, but the particular integral less so. There were some erroneous assumptions about the form of the particular integral.

(iii) Solutions were marred by accuracy errors in differentiation of the general solution and/or in simplification and collection of like terms following substitution. This led to a disappointing loss of marks by candidates who clearly knew the methods but who were unable to carry them out carefully.

(iv) Again, the method was known, but accuracy errors in execution were made by many candidates.

(v) For full marks, a statement was expected to the effect that, for sufficiently large t, the exponential term in each of the particular solutions for x and y tends to zero.